Eksamensnoter

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Contents

1 Lecture 1 - Introduction and Mathematical Preliminaries

In lecture one the following things are on the list of contents.

- What is elasticity?
- What is a continuum? The need for local quantities
- What is a tensor? Tensor algebra and index notation
- The eigenvalue problem (also see Matteos Misc. Notes)

1.1 Scalar and vector fields

1.1.1 Definitions

A scalar quantity expressed as a function of the Cartesian coordinates such as

$$f(x_1, x_2, x_3) = \text{constant}$$

Is a scalar field. A scalar field only contains a magnitude - it is a function associating a single number to each point in a region of space. An example of a scalar field could be the temperature on a plate.

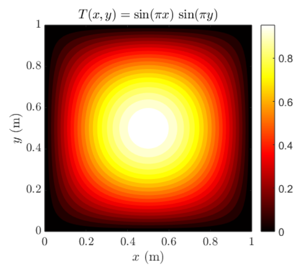


Figure 1: Temperature field on plate

A similar quantity is a vector field. A vector field contains both a direction and a magnitude - a vector field assigns a vector to each point in space. An example could be the velocity of a particle.

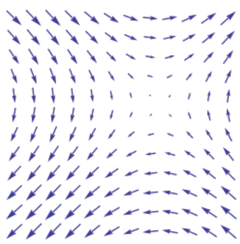


Figure 2: Example of a Vector field

1.1.2 Gradient

The gradient of a scalar field is defined as

$$\mathbf{grad} f = \nabla f = \frac{\partial f}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial f}{\partial x_3}\,\mathbf{e}_3 = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\Big).$$

Here $\mathbf{grad} f$ is a vector point function which is orthogonal to the surface $f = \textit{constant}$, everywhere. One may find the components by the appropriate dot product as

$$\frac{\partial f}{\partial x_1} = \mathbf{e}_1 \cdot \nabla f. \tag{1}$$

1.1.3 Operators

The del operator ∇ may be treated as a vector

$$\nabla(\ ) = \frac{\partial(\ )}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial(\ )}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial(\ )}{\partial x_3}\,\mathbf{e}_3.$$

For higher orders this may be written as

$$\nabla^2() = \nabla\cdot\nabla() = \frac{\partial^2()}{\partial x_1^2} + \frac{\partial^2()}{\partial x_2^2} + \frac{\partial^2()}{\partial x_3^2}.$$

These are scalar quantities, and are used frequently.

1.1.4 Divergence

The divergence of a vector field uses the del operator, and is defined as

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}.$$

Commonly writtwen as $\Delta\mathbf{A}$.

1.1.5 Curl

Dont really use this, but since there are two ways to mulitply vectors there are also two derivative forms. The curl is the other one, find it on page 5 of the book.

1.2 Integral Theorems

In this course, two integral theorems are of particular interest. These are used to transform between contour, area and volume integral. They are Green's theorem and the divergence theorem.

1.2.1 Green's theorem

Consider two functions $P(x, y)$ and $Q(x, y)$, which are continuous and have continuous first partial derivatives in a two-dimensional domain D. Greens theorem states that

$$\oint_C \big(P\,dx + Q\,dy\big) = \iint_A \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy,$$

Where A is closed region within D bounded by C. Its a way to relate the line integral of a closed curve to a double integral.

1.2.2 Divergence Theorem

Once again considering the function $P(x, y)$ and $Q(x, y)$, we consider a continously differentiable vectorpoing function $\mathbf{G}$ (essentially the same as a vectorfield) in the domain D. The divergence theorem states that

$$\int_V (\nabla \cdot \mathbf{G})\, dV = \int_A (\mathbf{n} \cdot \mathbf{G})\, dA.$$

Here $\nabla \cdot \mathbf{G}$ is the divergence. V is the volume bounded by the oriented surface A and $\mathbf{n}$ is the positive normal to A. Its purpose is essentially to convert hard surface integrals into easier volume integrals.

1.3 Indicinal Notation

Indicinal notation is a way to handle lengthy equations. Forexample, the cartesion vector $\mathbf{A}$ may be written in terms of its components A_i, where the index i takes the values $1, 2, 3$.

1.3.1 Einstein summation convention

If any subscript appears twice, we sum over it. So for example

$$A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Or

$$D_{jj} = D_{11} + D_{22} + D_{33}.$$

So there are two types of indices. There is a free index, which appears only once in each term of the equation and ranges from 1 to 3. The other type appears twice in a single term and is then summed over, this is known as a dummy index

1.3.2 Partial differentiation in indicinal notation

We denote partial differentiation using a comma

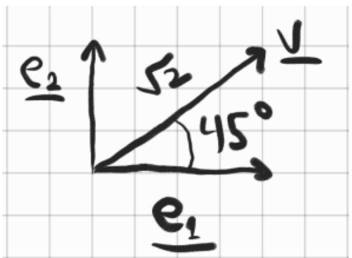
$$\frac{\partial A_i}{\partial x_j} = A_{i,j}$$

In which case both i and j are free indices, so this represents 9 quantities. If the indices are repeated, we get the divergence

$$\Delta A = \frac{\partial A_i}{\partial x_i} = A_{i,i} = A_{1,1} + A_{2,2} + A_{3,3}.$$

1.3.3 Examples of Vectors and indicinal notation

Consider the vector in the figure



We may write the vector in terms of its components as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e_2}$$

If we ever want to get a component of the vector, say forexample v_1, we can simply take the dot product with the corresponding dot product

$$v_1 = \mathbf{v} \cdot \mathbf{e}_1 = (1,1) \cdot (1,0) = 1$$

1.4 Coordinate Rotation

1.4.1 Rotation of coordinate system

This is very used when dealing with principal stresses and strains. I will explain using the figure from the book, it helps a lot to follow the figure and go along.

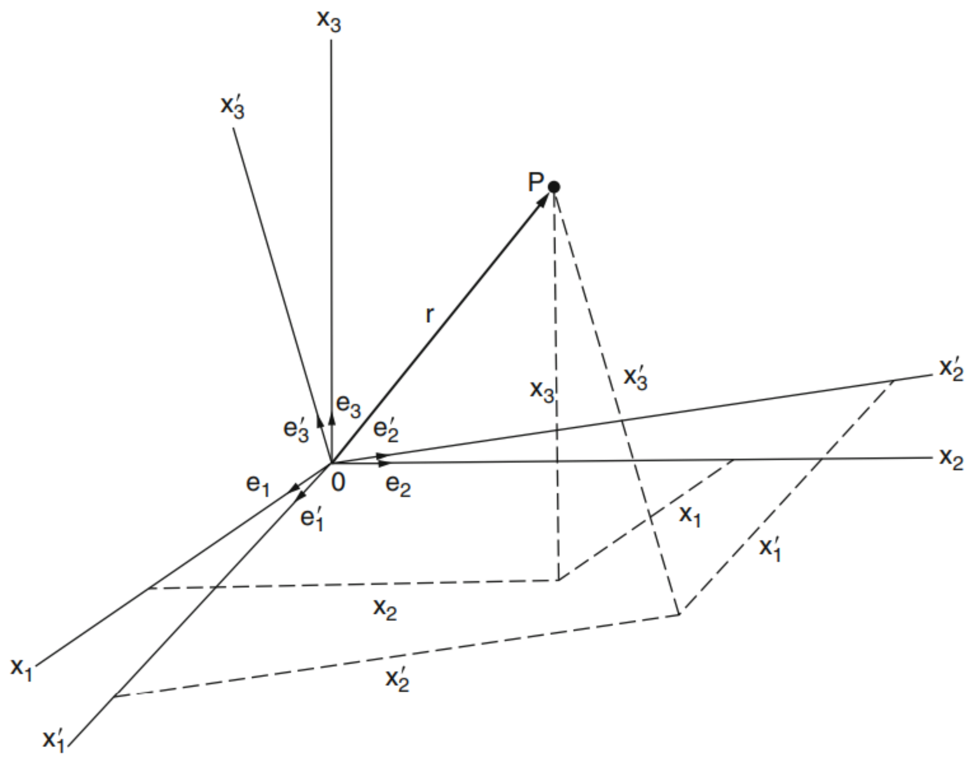


Figure 3: Two cartesian reference frames

In the figure we have a point P and a position vector pointing to that point $\mathbf{r}$. Along with this we have two different sets of cartesian reference frames, x_i and x_i' who both share the same origin. The corresponding unit vectors are $\mathbf{e}_i$ and $\mathbf{e}_i'$. First consider the point P in the x_i reference frame, it has coordinates $P(x_1, x_2, x_3)$. In the primed reference grame, we instead have $P(x_i') = P(x_1', x_2', x_3')$. The point is the same, but its just described in two different ways - we wish to find a way to translate between the two reference points. We do this by describing on reference frame in terms of the other, so lets start with describing x'_1 in terms of x_i - to aid i have provided a figure below. Its essentially just breaking down a 3D vector into its components using coordinate direction angles (like we did in statics)

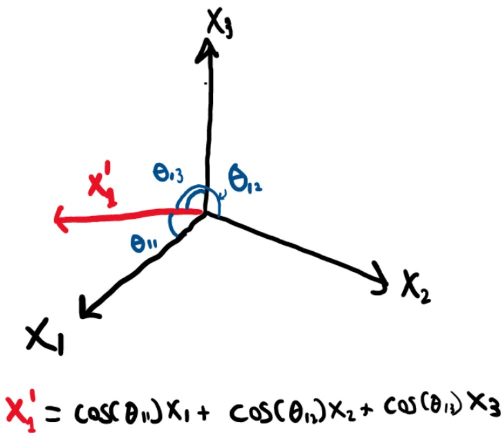


Figure 4: Breaking x'_1 into components in x_i reference frame

Now using the definition that

$$\alpha_{ij} = \cos(x'_i, x_j) = \mathbf{e}'_i \cdot \mathbf{e_j} = \cos(\mathbf{e}'_i, \mathbf{e}_j)$$

Where α_{ij} is the direction cosine and is the cosine of the angle between the i'th primed and the j'th unprimed axis We may sate that for x'_1 we get

$$x'_1 = \alpha_{11}\, x_1 + \alpha_{12}\, x_2 + \alpha_{13}\, x_3$$

Similarly for the other two axes:

$$\begin{aligned} x'_2 &= \alpha_{21}\, x_1 + \alpha_{22}\, x_2 + \alpha_{23}\, x_3, \\ x'_3 &= \alpha_{31}\, x_1 + \alpha_{32}\, x_2 + \alpha_{33}\, x_3. \end{aligned}$$

For convenience, this may be written in matrix form as

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

$$\boldsymbol{x'} = \boldsymbol{R}\boldsymbol{x}$$

Alternatively this can be written in indicinal notation as

$$x'_i = \alpha_{ij} x_j$$

1.4.2 Rotation of tensors

To change a tensor from one reference frame to the other, consider consider the tensor $\boldsymbol{T}$. To go from the unprimed coordinate system to the primed, use the following formula:

$$\boxed{\boldsymbol{T'} = \boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^{\boldsymbol{T}}}$$

Alterntively this can be written as

$$\boldsymbol{T}' = [\alpha_{ij}]\boldsymbol{T}[\alpha_{ji}]$$

Or in index notation

$$T'_{ij} = \alpha_{ik}\alpha_{jl}T_{kl}$$

This is often used to rotate strain or stress tensors.
If one wants to go the other way around, then

$$\boldsymbol{T} = \boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{T}'\boldsymbol{R}$$

1.5 Operational Tensors (Kronecker delta and permutation tensor)

1.5.1 Kronecker delta

The Kronecker delta is defined such that

$$\delta_{ij} = 1 \text{ if } i = j$$
$$\delta_{ij} = 0 \text{ if } i \neq j.$$

Another definition is simply

$$\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j.$$

It is very useful, as it can be used to swap on indice with another, forexample

$$v_i\delta_{ij} = v_j.$$

so ij swaps i with j

1.5.2 Permutation tensor

The permutation tensor ε_{ijk} has the following properties:

$\varepsilon_{ijk} = 0$	If any two of i, j, k are equal
$\varepsilon_{ijk} = 1$	For an even permutation (forward on the number line $1, 2, 3, 1, 2, 3, \dots$)
$\varepsilon_{ijk} = -1$	For an odd permutation (backward on the number line $3, 2, 1, 3, 2, 1, \dots$)

So examples could be $\varepsilon_{112} = 0$ $\varepsilon_{312} = 1$ and $\varepsilon_{321} = -1$

Its real use case comes when evaluating the cross product. A lengthy derivation is in the book, but essenctially if one wants the cross product

$$\mathbf{A} \times \mathbf{B}$$

In component form, then this may be written as

$$\varepsilon_{ijk}A_jB_k.$$

Useful later for conservation of angular momentum. The derivation is on page 12 of book.

1.6 Tensors

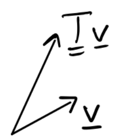
A tensor is a set which transform from one coordinate system to another. In theory of elasticity, we use 0,1 and 2nd order tensors. 0 order is a scalar, 1st order is a vector, and 2nd order is a matrix. It is a mathematical object that linearly relates vectors (or other tensors) in one coordinate system to vectors (or tensors) in another, The difference between a Tensor and a matrix is that a matrix can be non-physical, whereas a tensor has to describe a physical state.

A tensor takes a vector and gives another vector

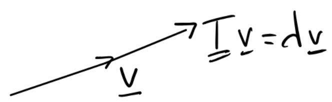
$$\mathbf{w} = \boldsymbol{T}\mathbf{v}$$

1.7 Eigenvalues and Eigenvectors

Consider a tensor $\boldsymbol{T}$ and a vector $\mathbf{v}$. In generel, the vector $\mathbf{w} = \boldsymbol{T}\mathbf{v}$ is not parralel to $\mathbf{v}$. If it is, then $\mathbf{v}$ is an eigenvector of $\boldsymbol{T}$.



$\underline{v}$ is not the eigenvector.



$\underline{v}$ is an eigenvector.

When $\mathbf{v}$ is an eigenvector, the tensor $\boldsymbol{T}$ does not change its direction, but only scales it by some value - this value is the eigenvalue λ. In other words, the eigenvalue λ is the factor $\boldsymbol{T}$ scales the eigenvector $\mathbf{v}$.

We find eigenvectors by rewriting:

$$\boldsymbol{T}\mathbf{v} = \lambda\mathbf{v} \Rightarrow (\boldsymbol{T} - \lambda\boldsymbol{I})\mathbf{v} = 0$$

To avoid the trivial solution $\mathbf{v} = \mathbf{0}$ we require that

$$\det(\boldsymbol{T} - \lambda\boldsymbol{I}) = 0$$

By solving this equation the eigenvalues λ_i are found. To find the eigenvectors solve

$$(\boldsymbol{T} - \lambda_i\boldsymbol{I})\mathbf{v}_i = 0$$

For $i = 1, 2, 3$ (no summation)

1.8 Examples and homework problems

1.8.1 Example 1

Show that $\delta_{ij}\delta_{jk} = \delta_{ik}$ Expanding using the summing convention yields

$$\delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{k3}$$

So for any choice of i and k this yields δ_{ik}. As an example lets choose $i = 2$

$$\delta_{21}\delta_{1k} + \delta_{22}\delta_{2k} + \delta_{23}\delta_{k3} = \delta_{2k}$$

1.8.2 HW 1.1 (a) and (e)

(a) show that $\delta_{ij}\delta_{ij} = 3$

$$\delta_{ij}\delta_{ij} = \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} + \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23} + \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33}$$

$$\Downarrow$$

$$\delta_{ij}\delta_{ij} = (1)(1) + (0)(0) + (0)(0) + (0)(0) + (1)(1) + (0)(0) + (0)(0) + (0)(0) + (1)(1) = 3$$

Alternatively this could have been shown using the fact that δij swaps the indices i and j

$$\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

(e) show that $\delta_{ij}C_{jkl} = C_{ikl}$

$$\delta_{ij}C_{jkl} = \delta_{i1}C_{1kl} + \delta_{i2}C_{2kl} + \delta_{i3}C_{3kl}$$

If i choose as an example $i = 1$, so i would get C_{1kl}, we can check if it holds using the equation

$$\delta_{1j}C_{jkl} = \delta_{11}C_{1kl} + \delta_{12}C_{2kl} + \delta_{13}C_{3kl} = C_{1kl}$$

This could also have been shown by using the fact that δij swaps the indices i and j

1.8.3 Write the divergence theorem defined in (1.14) in indicinal notation

From (1.14) in the book we had

$$\int_V \big(\nabla \cdot \mathbf{G}\big)\, dV = \int_A \big(\mathbf{n} \cdot \mathbf{G}\big)\, dA.$$

The divergence on the RHS in indicinal notation may be written as $G_{i,i}$ and the dot product on the LHS as n_iG_i, thus yielding

$$\int_V G_{i,i} dV = \int_A n_i G_i dA.$$

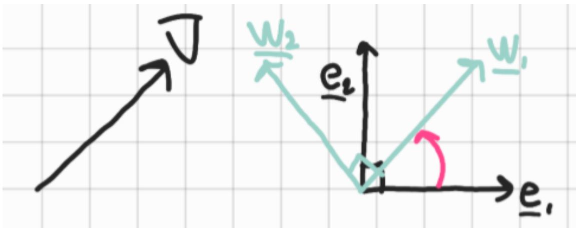
2 Lecture 2: Stress part 1

In this lecture the following things are on the list of contents.

- The eigenvalue problem: Change of reference frame
- How are forces handled by a solid?
- The stress vector and the Cauchy stress tensor
- The Cauchy theorem: Stress on a normal plane and dyadic represenation.

2.1 Change of reference frame

Consider a vector and two different reference frames



We may write the vector in terms of its components in either reference frame.

$$\mathbf{v} = \underbrace{v_i e_i}_{e_1, e_2\text{-frame}} = \underbrace{v_i' w_i}_{w_1, w_2\text{-frame}}$$

Remember! $v_i \neq v_i'$ since we're in two different frames. If we take the equation above, and multiply both sides by w_j we get

$$v_i e_i \cdot w_j = v_i' w_i \cdot w_j$$

Now using that $e_i \cdot w_j = \alpha_{ij}$ and $w_i \cdot w_j = \delta_{ij}$ the equation becomes

$$v_i \alpha_{ij} = v_i' \delta_{ij}$$

Swapping LHS and RHS aswell as indices due to kronecker delta yields

$$\boxed{v_j' = \alpha_{ji} v_i}$$

Now we have a formula for calculating vector components in new coordinate system based on the old one! Just choose j. We may write this in matrix form as

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Using the fact that $\alpha_{ij} = e_i \cdot w_j$ we may rewrite this as

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \underbrace{\begin{bmatrix} w_1 \cdot e_1 & w_1 \cdot e_2 & w_1 \cdot e_3 \\ w_2 \cdot e_1 & w_2 \cdot e_2 & w_2 \cdot e_3 \\ w_3 \cdot e_1 & w_3 \cdot e_2 & w_3 \cdot e_3 \end{bmatrix}}_{R} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Since e_i is our reference coordinate system, then the rows of the rotation matrix above, is simply just the new coordinate axis in our old reference system, i.e

$$\boldsymbol{R} = \begin{bmatrix} - - w_1 - - \\ - - w_2 - - \\ w_3 \end{bmatrix}$$

or in other words

$$\mathbf{v}' = \boldsymbol{R}\mathbf{v}$$

Also note that

$$\mathbf{w}_i = \boldsymbol{R}^T \mathbf{e}_i$$

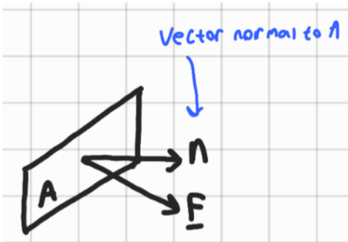
(se noter under lecture 2 for eksampler) Another important property is that

$$\boldsymbol{R^T R = I}$$

i.e they are orthogonal.

2.2 State of stress

To determine stress, we need both a surface and a force - this is very important. Never one without the other. Consider a surface with area A and unit normal vector n on which there is an applied load $\mathbf{F}$



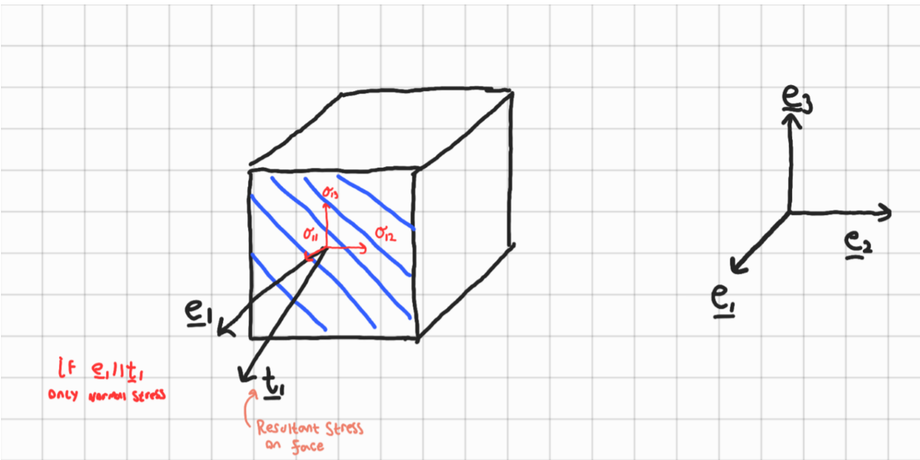
An estimate of the traction / stress vector on the surface is given as

$$\mathbf{t}_n = \frac{\mathbf{F}_n}{A_n}$$

The actual traction vector is given by taking the limit of this as $A \to 0$

$$\boxed{\mathbf{t}_n = \lim_{A_n \to 0} \frac{\mathbf{F}_n}{A_n}}$$

The traction vector $\mathbf{t}_n$ is the stress intensity at the point for the particular orientation of the are element specified by $\mathbf{n}$. A complete description at the point requires that the state of stress is known in all directions, so the traction vector is not enough by itself. We need to find a relationship between stress and $\mathbf{n}$, we look at a small cube of our material and consider the surface with unit normal $\mathbf{e}_1$



The traction vector on this face may be written in terms of its cartesian components

$$\mathbf{t}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3$$

This applied to all the three different sides, yielding the general relation

$$\boxed{\mathbf{t_i} = \boldsymbol{\sigma}_{ij}\mathbf{e}_j}$$

Set i component to desired surface and sum over j. The coefficients $\sigma_{11}, \sigma_{12}..\sigma_{33}$ are stresses, and the entire array of these stresses forms the **Cauchy stress tensor**. The subscripts are as follows for the stresses

- i is the surface on which the stress acts
- j is the direction in which the stress actors

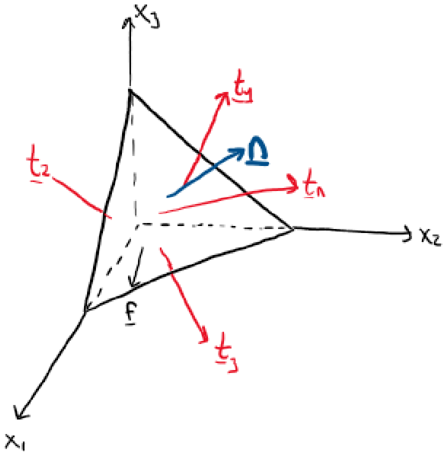
2.3 Stress at a point

Before we looked at the stress on one of the planes of the cube, but what if we want the stress in an aribtrary direction? In this case, the formula is:

$$t_i = \sigma_{ij}n_j$$

You pick i=1,2,3 depending on which component you want, and then sum over j. I'm going to derive this by hand. This is the cauchy theorem

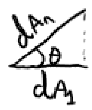
We consider a general plane. We look at a tetrahedon with height h



We have $\underline{t}_n$ on the plane of interest along with $\underline{t}_i$ on the planes with unit normals $-\underline{e}_i$.
Since the body is in equilibrium $\Sigma \underline{F} = 0$. The forces are givn as $\underline{t}_i dA_i$, $i=1,2,3$ and $\underline{t}_n dA_n$.
Equilibrium yields (No summation over n!)

$$\underline{t}_n dA_n - \underline{t}_i dA_i - \underline{f}\left(\tfrac{1}{3} dA_n h\right)$$

We want to relate the areas.



$$\rightarrow dA_1 = \cancel{dA_n \cos\theta} = dA_n \underline{e}_1 \cdot \underline{n} = n_1$$

inserting this

$$\underline{t}_n dA_n - \underline{t}_i n_i dA_n - \underline{f}\left(\tfrac{1}{3} dA_n h\right) = 0$$

Factoring

$$\left(\underline{t}_n - \underline{t}_i n_i - \underline{f}\tfrac{h}{3}\right) dA_n = 0$$

as $h \to 0$,

$$\left(\underline{t}_n - \underline{t}_i n_i\right) dA_n = 0$$

We Resolve $\underline{t}_n$ into its components $t_i e_i$. For convenience we swap dummy indices of $\underline{t}_i n_i \to \underline{t}_j n_j$
The Equation is satisfied if

$$t_i \underline{e}_i = \underline{t}_j n_j$$

We know that $\underline{t}_j = \sigma_{ij} \underline{e}_i$

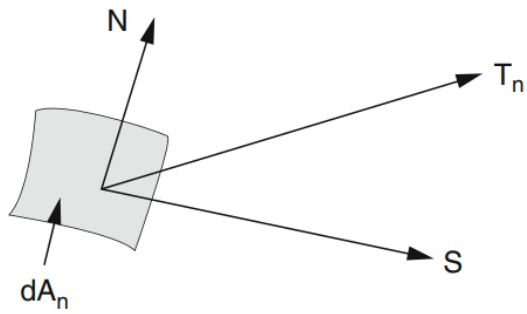
$$t_i \underline{e}_i = \sigma_{ij} \underline{e}_i n_j$$

we must then have

$$t_i = \sigma_{ij} n_j$$

or in vector form

$$\underline{t}_n = \underline{\sigma}\, \underline{n}$$



H

2.4 Stress on a normal plane

Consider the figure, on which we see are differential surface element with a traction vector. To get the normal component, simply do

$$\sigma_{nn} = \mathbf{t}_n \cdot \mathbf{n}$$

To get the shear do

$$\sigma_{ns} = \mathbf{t}_n \cdot \mathbf{s} = (t_i t_i - \sigma_{nn}^2)^{1/2}$$

2.5 Examples and homework problems

I have put the examples from this section together with the ones in lecture 4 for a more complete overview. The homework for from the lecture is here though.

2.5.1 HW 1.1(c)

Show that $\alpha_{ij} b_{jk} = \alpha_{in} b_{nk}$

$$\alpha_{ij} b_{jk} = \alpha_{i1} b_{1k} + \alpha_{i2} b_{2k} + \alpha_{i3} b_{3k} = \alpha_{in} b_{nk}$$

I mean j is a dummy index, i can call it whatever lol.

2.5.2 HW 2.1

Find the components of the traction vector

$$\mathbf{t} = \boldsymbol{\sigma}\,\mathbf{n}, \qquad \text{where} \quad \mathbf{n} = (n_1, n_2, n_3) = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right),$$

for the following stress states: (there is a typo in book)

1. $\sigma_{11} = \sigma, \quad \sigma_{12} = 0, \quad \sigma_{13} = 0,$
   $\sigma_{21} = 0, \quad \sigma_{22} = \sigma, \quad \sigma_{33} = \sigma.$

2. $\sigma_{11} = \sigma, \quad \sigma_{12} = \sigma, \quad \sigma_{13} = 0,$
   $\sigma_{22} = \sigma, \quad \sigma_{23} = 0, \quad \sigma_{33} = 0.$

Part (a)

Could do this with the formula above, but im going to do this component wise, so using $t_i = \sigma_{ij} n_j$. Also recall that $\sigma_{ij} = \sigma_{ji}$

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = \frac{\sigma}{\sqrt{2}}$$

$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 = \frac{\sigma}{\sqrt{2}}$$

$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 = 0$$

Part (b) i will do using matrix formulation

$$\mathbf{t}_n = \boldsymbol{\sigma} \mathbf{n}$$

Yields

$$\begin{bmatrix} \sigma & \sigma & 0 \\ \sigma & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma\sqrt{2} \\ \sigma\sqrt{2} \\ 0 \end{pmatrix}$$

2.5.3 HW 2.6

At a given point in a body

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31}$$

Show that the normal stresses are equal to $-p$ and that the shearing stresses vanish for any other Cartesian coordinate system.

We may write

$$\boldsymbol{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\boldsymbol{I}$$

This is a case of hydrostatic stress. The entries on the diagonal are the normal stresses. To show that any Cartesian coordinate system has no shearing stresses, we can state the its the identity matrix scaled by something, so if we take an arbiraity rotaion matrix $\boldsymbol{R}$ and try to change our coordinate basis, we would just get

$$\boldsymbol{\sigma}' = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^T = \boldsymbol{R}(-p\boldsymbol{I})\boldsymbol{R}^T = -p\boldsymbol{R}\boldsymbol{R}^T = -p\boldsymbol{I} = \boldsymbol{\sigma}$$

The last part is because they are orthogonal, I.E $\boldsymbol{R}\boldsymbol{R}^{\boldsymbol{T}} = \boldsymbol{I}$. So no matter how we rotate it, we'll still have the original stress matrix, which has no shearing stress.

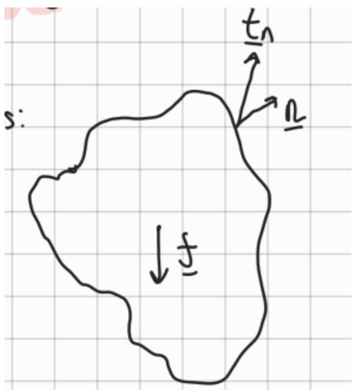
3 Lecture 3: Stress continued. Balance laws & principal stress

In this lecture we have the following bullet points

- A bit more on the cauchy theorem
- Equilibrium equations
- Principal stresses
- Stress transformation (also see notes on eigenvectors)

3.1 Equilibrium (Conservation of linear momentum)

Consider an arbitrary body under stress



Body forces are in units of $[\frac{N}{m^3}]$.

We do static equilibrium in integral form since we want to generalise

$$\sum \mathbf{F} = \mathbf{0} \Rightarrow \int_V \mathbf{f} dV + \int_A \mathbf{t}_n dA = \mathbf{0}$$

For convenience, we rewrite this using indicinal notation

$$\int_V f_i dV + \int_A t_i dA = 0$$

Using that $t_i = \sigma_{ji} n_j$ (we dont know its symmetric yet :))

$$\int_V f_i dV + \int_A \sigma_{ji} n_j dA = 0$$

To get this under one integral, we use the divergence theorem. Recall that $\int_V G_{i,i} dV = \int_A n_i G_i dA$.

$$\int_V f_i dV + \int_V \sigma_{ji,j} dV = 0$$

Factoring

$$\int_V (f_i + \sigma_{ji,j}) dV = 0$$

For this to hold we may state that

$$\boxed{f_i + \sigma_{ji,j} = 0}$$

Which is our equilibrium equation. In matrix notation it becomes

$$\mathbf{f} + \text{div}\boldsymbol{\sigma}^{\boldsymbol{T}} = 0$$

the stress tensor is symmetric, so we may remove the transpose, yieding the equations

$$\sigma_{ij,j} + f_i = 0$$

And

$$\text{div}\boldsymbol{\sigma}_{ij} + \mathbf{f} = 0$$

3.1.1 Quick note on Angular momentum

Its a very lengthy derivation, so it might not be on the exam - however there is a key take away. **The reason the stress tensor is symmetric is due to angular momentum / rotational equilibrium**

3.2 Principal stresses

A principal plane, is a plane on which the normal stress is at an extremum. The corresponding stress on this plane, which is then purely normal, is called the principal stress. With principal stresses we have no shear. The maximum principal stress, it the largest amount of normal stress the body can be subject to, if its oriented in the corresponding principal direction. Lets derive this how to find them.

If the traction vector $\mathbf{t}_n$ is to be purely normal stress, it has to be parallel to the unit normal vector defining the area $\mathbf{n}$. A figure is drawn for completeness

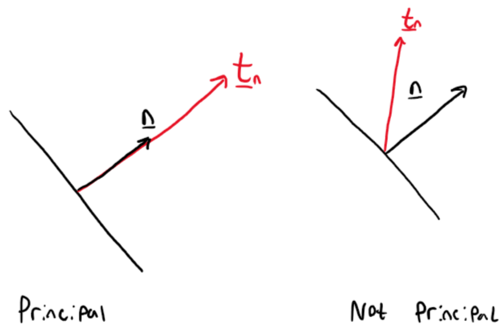


Figure 5: Principal vs not principal

To enforce that the traction vector is parralel to the normal vector, we may state the it simply has to be the normal vector scaled by some value λ.

$$\mathbf{t}_n = \lambda \mathbf{n}$$

From this we can derrive the eigenvalue problem. Dotting both sides by $\mathbf{e}_i$ to get it on component form

$$t_i = \lambda n_i$$

We know that $t_i = \sigma_{ij} n_j$, inserting on the LHS

$$\sigma_{ij} n_j = \lambda n_i$$

Using the indice swapping of the kronecker delta, we may state that $n_i = \delta_{ij} n_j$.

$$\sigma_{ij} n_j = \lambda \delta_{ij} n_j$$

Subtracting and factoring yields

$$(\sigma_{ij} - \delta_{ij}\lambda) n_j = 0$$

This is the eigenvalue problem. In matrix form we get

$$(\boldsymbol{\sigma} - \lambda \boldsymbol{I})\mathbf{n} = \mathbf{0}.$$

Solving this in 3D yields 3 eigenvalues, which are the three principal stresses, aswell as the corresponding eigenvectors $\mathbf{n}$ which are the principal directions.

3.3 Stress invariants

Two are important. The determinant, which we know, and then the trace, which is just the sum of the elements on the diagonal. Invariants do not change under rotation - important properties

3.4 Properties and Special States of stress

i will fill this in later, look at lecture 1-4 for examples i wrote down in samsung notes.

3.5 Homework

3.5.1 HW 2.4

Determine the body forces for which the following stress field describes a state of equilibrium

$$\begin{aligned} \sigma_{xx} &= -2x^2 - 3y^2 - 5z, & \sigma_{xy} &= z + 4xy - 6, \\ \sigma_{yy} &= -2y^2 + 7, & \sigma_{xz} &= -3x + 2y + 1, \\ \sigma_{zz} &= 4x + y + 3z - 5, & \sigma_{yz} &= 0. \end{aligned}$$

Will use the equilibrium equation $\sigma_{ij,j} + f_i = 0$

$$\begin{aligned} f_x - 4x + 4x + 0 = 0 &\Rightarrow f_x = 0 \\ f_y + 4y - 4y + 0 = 0 &\Rightarrow f_y = 0 \\ f_z - 3 + 0 + 3 &\Rightarrow f_z = 0 \end{aligned}$$

The body forces are required to be 0 for the stress field to describe a state of equilibrium.

3.5.2 HW 2.5

Determine whether the following stress field is admissible in an elastic body when body forces are negligble

$$\boldsymbol{\sigma} = \begin{bmatrix} yz+4 & z^2+2x & 5y+z \\ z^2+2x & xz+3y & 8x^2 \\ 5y+z & 8x^2 & 2xyz \end{bmatrix}$$

Will once again use the equlibrium equation div$\boldsymbol{\sigma} + \mathbf{f} = 0$

$$f_x = 0 + 0 + 1 = 0 \Rightarrow f_x = -1$$

Can already tell here what it is not admissible, as $\mathbf{f} \neq \mathbf{0}$

3.5.3 HW 2.9

The stress tensor at a point $P(x, y, z)$ in a solid is

$$\boldsymbol{\sigma} = \begin{bmatrix} 120 & 40 & 30 \\ 40 & 150 & 20 \\ 30 & 20 & 100 \end{bmatrix}$$

(a) Determine the stresses with respect to a set of axes x',y' and z' for which the matrix of direction cosines is

$$[\alpha_{ij}] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For this we simply do

$$\boldsymbol{\sigma}' = [\alpha_{ij}]\boldsymbol{\sigma}\alpha_{ji} = \begin{bmatrix} 107.9 & -33.0 & -2.3 \\ -33 & 162.1 & 36.0 \\ -2.3 & 36.0 & 100.0 \end{bmatrix}$$

(b) Compute the principal stresses using both the x, y, z basis and the x', y', z' basis.
I do this using MatLab. In x, y, z basis i get $\lambda_1 = 77.17, \lambda_2 = 102.42, \lambda_3 = 190.4$. I get the same in the primed basis, which must mean they are invariant. This makes sense, the principal stresses should stay the same, we just rotated our frame - we should arrive at the same principal stresses regardless of what our start basis was.

(c) Find the orientation of the principal planes in the x, y, z basis.
Will be done using $(\boldsymbol{\sigma} - \lambda^{(i)}\boldsymbol{I})\mathbf{n}^{(i)}$ for $i = 1, 2, 3$. This yields

$$\mathbf{n}^{(1)} = (-0.66, 0.16, 0.73)$$
$$\mathbf{n}^{(2)} = (0.48, -0.65, 0.59)$$
$$\mathbf{n}^{(3)} = (0.57, 0.74, 0.35)$$

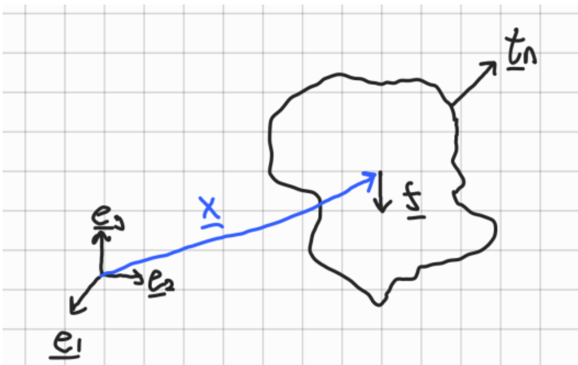
4 Lecture 4: Stress part III - Angular momentum

The bullet points for this lecture are

- Principle of angular momentum
- More on principal stresses and stress transformation

4.1 Angular momentum

We consider a body equilibrium.



Now summing torques, as this has to be in equilibrium, we know that to calculate the torque at a point, we may take the cross product between that point and origin, and the force. Again we want to keep it in a general case, so we use integrals. We get

$$\int_V \mathbf{x} \times \mathbf{f} dV + \int_A \mathbf{x} \times \mathbf{t}_n dA = 0$$

We want to write this on component form, we use the permutation tensor as discussed in 1.5.2 we may write the cross products as

$$\mathbf{x} \times \mathbf{f} = \varepsilon_{ijk} x_j f_k$$
$$\mathbf{x} \times \mathbf{t}_n = \varepsilon_{ijk} x_j t_k = \varepsilon_{ijk} x_j \sigma_{lk} n_l$$

Inserting this into the original equatio

$$\int_V \varepsilon_{ijk} x_j f_k dV + \int_A \varepsilon_{ijk} x_j \sigma_{lk} n_l dA = 0$$

We wish to use the divergence theorem to put them under a common integral.

$$\int_V \varepsilon_{ijk} x_j f_k dV + \int_V (\varepsilon_{ijk} x_j \sigma_{lk})_{,l} dV = 0$$

We'll carry out the derivative.

$$(\varepsilon_{ijk} x_j \sigma_{lk})_{,l} = \varepsilon_{ijk} (x_j \sigma_{lk})_{,l} = \varepsilon_{ijk} (x_{j,l} \sigma_{lk} + x_j \sigma_{lk,l})$$

Its apparent, that $x_{j,l} = 1$ if $j = l$, otherwise its 0. I.E its simply δ_{jl}. Inserting all this newfound knowledge

$$\int_V \varepsilon_{ijk}(x_j f_k + \delta_{jl}\sigma_{lk} + x_j \sigma_{lk,l})dV = 0$$

We may do a bit of factoring, and also change indices due to the kronecker delta

$$\int_V \varepsilon_{ijk}(x_j(f_k + \sigma_{lk,l}) + \sigma_{jk})dV$$

The term $(f_k + \sigma_{lk,l})$ is simply the linear equilibrium equation, this equates to 0 if there is equilbrium, so by removing that we end up with

$$\int_V \varepsilon_{ijk}\sigma_{jk}dV = 0$$

This equation holds if

$$\varepsilon_{ijk}\sigma_{jk} = 0$$

This equation shows that the strain tensor is symmetric. By evaluating for $i = 1, 2, 3$. I will show for $i = 1$

$$\varepsilon_{1jk}\sigma_{jk} = \varepsilon_{111}\sigma_{11} + \varepsilon_{112}\sigma_{12} + \varepsilon_{113}\sigma_{13} + \varepsilon_{121}\sigma_{21} + \varepsilon_{122}\sigma_{22} + \varepsilon_{123}\sigma_{23} + \varepsilon_{131}\sigma_{31} + \varepsilon_{132}\sigma_{32} + \varepsilon_{133}\sigma_{33} = 0$$

$$\Downarrow$$

$$\varepsilon_{123}\sigma_{23} + \varepsilon_{132}\sigma_{32} = 0 \Rightarrow \sigma_{23} - \sigma_{32} = 0$$

Finally yielding

$$\sigma_{23} = \sigma_{32}$$

4.2 Examples and Homework (could be put together with examples in lec 3)

4.2.1 Examples

4.2.2 HW 2.2 (only a, b ,d and e)

The state of stress at a point P in a structure is given by

$$\sigma_{11} = 20000, \sigma_{22} = -15000, \sigma_{33} = 3000$$
$$\sigma_{12} = 2000, \sigma_{23} = 2000, \sigma_{31} = 1000$$

(a) Compute the scalar components of the traction $\mathbf{t}_n$ on the plane passing through P whose outward normal vector $\mathbf{n}$ makes equal angles with the coordinate axis. Such a unit normal vector must be

$$\mathbf{n} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$$

The stress tensor is

$$\boldsymbol{\sigma} = \begin{bmatrix} 20000 & 2000 & 1000 \\ 2000 & -15000 & 2000 \\ 1000 & 2000 & 3000 \end{bmatrix}$$

Using $\mathbf{t}_n = \boldsymbol{\sigma}\mathbf{n}$

$$\mathbf{t}_n = \begin{bmatrix} 20000 & 2000 & 1000 \\ 2000 & -15000 & 2000 \\ 1000 & 2000 & 3000 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 13279.06 \\ -6350.85 \\ 3464.10 \end{pmatrix}$$

(b) Compute the normal and tangential components of stress on this plane.

Will be done using $\sigma_{nn} = \mathbf{t}_n \cdot \mathbf{n}$ and $\sigma_{ns} = \mathbf{t}_n - (\mathbf{t}_n \cdot \mathbf{n})\mathbf{n}$. Yielding

$$\sigma_{nn} = 6000$$

$$\sigma_{ns} = (9814.95, -9814.95, 0)^T$$

Something weird here
(d) Determine the principal stresses.
These are just the eigenvalues. Using MatLab or whatever yields

$$\lambda_1 = -15319, \lambda_2 = 3133.4, \lambda_3 = 20186.3$$

(e) Determine the direction of maximum principal stress
This is just the eigenvector corresponding to λ_3. This is found to be

$$\mathbf{v_3} = (0.996, 0.060, 0.065)^T$$

4.2.3 HW 2.3

The state of stress at a point is given by the components of stress tensor σ_{ij}. A plane is defined by the direction cosines of the normal $(1/2, 1/2, 1/\sqrt{2})$. State the general conditions for which the traction on the plane has the same direction as the x_2-axis and a magnitude of 1.0.
A lot to digest, we have that

$$\mathbf{n} = (1/2, 1/2, 1/\sqrt{2})$$

We require that

$$\mathbf{t}_n = (0, 1, 0).$$

We have that

$$\mathbf{t}_n = \boldsymbol{\sigma}\mathbf{n}$$

We may set this up in a vector form

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

From this we get

$$\begin{aligned} 0.5\sigma_{11} + 0.5\sigma_{12} + \sigma_{13}/\sqrt{2} = 0 \\ 0.5\sigma_{12} + 0.5\sigma_{22} + \sigma_{23}/\sqrt{2} = 0 \\ 0.5\sigma_{13} + 0.5\sigma_{23} + \sigma_{33}/\sqrt{2} = 0 \end{aligned}$$

These are the conditions to be met.

4.2.4 HW 2.8 (a) and (b)

The stress tensor at a point $P(x, y, z)$ in a solid is

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

(a) Show that the principal stresses are 4, 1, and -2.
I will just do the eigenvalue problem

$$\det(\boldsymbol{\sigma} - \lambda \boldsymbol{I}) = 0$$

$\Downarrow$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 4) - (-\lambda - 2) + (2 + \lambda) = -\lambda^3 + 3\lambda^2 + 6\lambda - 8 = 0$$

$$\Rightarrow \lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 4$$

(b) Find the orientation of the principal planes
Will be done using $(\boldsymbol{\sigma} - \lambda \boldsymbol{I})\mathbf{v} = \mathbf{0}$ for each λ

$$\mathbf{v}^{(1)} = (0, 1/\sqrt{2}, 1/\sqrt{2})^T, \mathbf{v}^{(2)} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T, \mathbf{v}^{(3)} = (0.817, 0.408, 0.408)^T$$

5 Lecture 5: Strain part I

Bullet points from this lecture are

- How do we describe the deformation of a continuum?
- How do we define strain?
- The strain tensor (covered in Lecture 6)

5.1 Introduction to strain

Displacement with respect to a coordinate system bay be physically observed, calulated or measured for a deformed elastic body. There is two components to displacement - one due to relative movements or distortions within the body (strains), and the other where the entire body moves (rigid body motion). The relationship between displacement and the internal distortions are known as the kinematric or the strain-displacement equations.

In general we have two kinds of strain. Normal strain: Change in length. Shear strains: Change in angle.

5.2 Strain

We consider the undeformed and the deformed positions of an elastic body shown. We designate two sets of coordinates x_i and X_i, called initial coordinated and final coordinates. We designate a reference point $P(x_i)$ on the undeformed body, and reference point $p(X_i)$ on the deformed body. We also locate neighbouring points $Q(x_i + dx_i)$ and $q(X_i + dX_i)$ that are separated from P and p by the distances ds and dS.

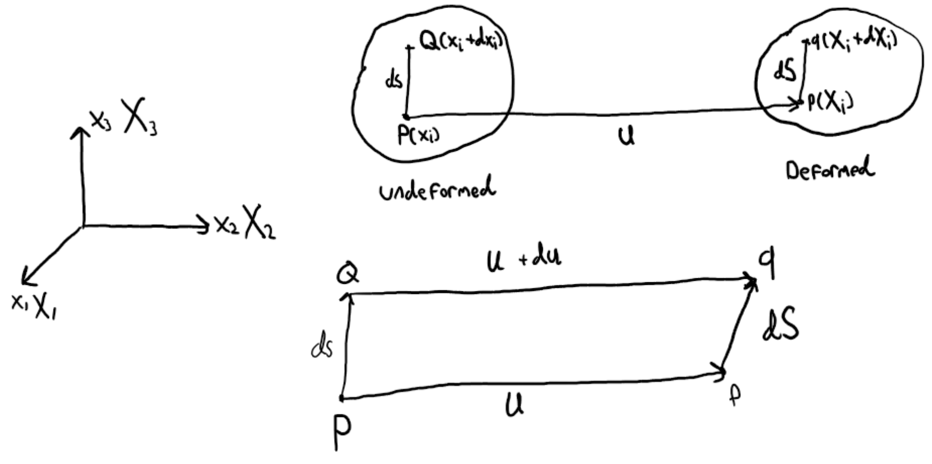


Figure 6: Displacement vector in Cartesian space

From this fugure, its apparent that

$$ds + (u + du) = u + dS \Rightarrow du = dS - ds$$

In the case where $dS = ds$ then $du = 0$. This means that there is only rigid body motion, and the displacement is homogeneous. We reduce to a 1D example to get a better understanding.

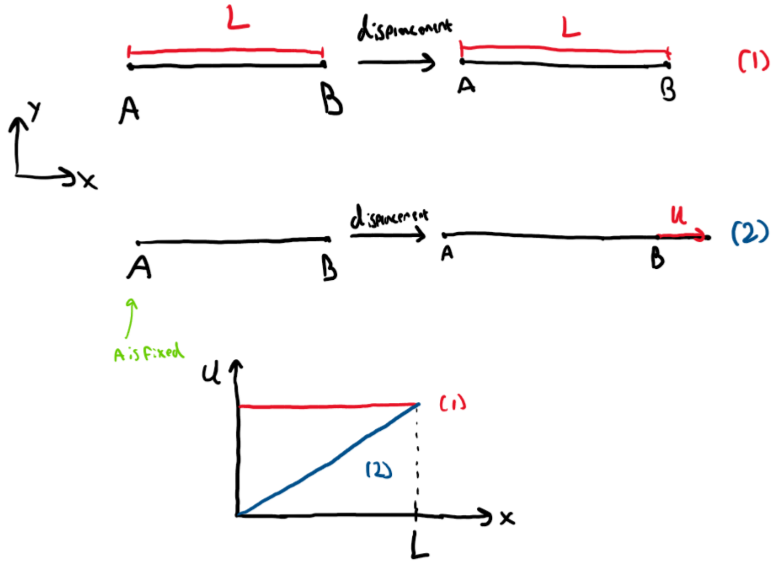


Figure 7: 1D example

In the figure we have two kinds of displacement. In the first one (1), no ends are fixed, and both end are displaced by the same amount, this is shown with the red line in the displacement vs x graph. In the other case (2), the left end is fixed, so when the bar is displaced, the displacement varies linearly across the bar. In this case (2) we have strain, where as in case 1 we have only displacement. From our definition of engineering strain we know that (since L is elongated by the displacement u

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{u}{L} = \frac{\partial u}{\partial x}.$$

We return to the 3D case in Figure (6). Using Pythagoras thereom to get $(dS)^2 = dX_i dX_i$ and $(ds)^2 = dx_i dx_i$ we get

$$(dS)^2 - (ds)^2 = dX_i dX_i - dx_i dx_i$$

We remember that $X_i(x_1, x_2, x_3)$ is a function of x_i. If we change our point x_i we change X_i, from the chain rule this means that $dX_i = \frac{\partial X_i}{\partial x_j} dx_j$. Conveniently for us, the first part is simply the divergence of X_i when we sum over j, so we may state that $dX_i = X_{i,j} dx_j$. We have two cases, so we introduce another index k, so we get

$$(dS)^2 - (ds)^2 = X_{i,j} dx_j X_{i,k} dx_k - dx_i dx_i$$

We do a trick, and write $dx_i dx_i = \delta_{ij}\delta_{ik} dx_j dx_k$ and get

$$(dS)^2 - (ds)^2 = X_{i,j} dx_j X_{i,k} dx_k - \delta_{ij}\delta_{ik} dx_j dx_k$$

This allows us to factor

$$(dS)^2 - (ds)^2 = (X_{i,j} X_{i,k} - \delta_{ij}\delta_{ik}) dx_j dx_k$$

Cleaning up the kronecker delta

$$(dS)^2 - (ds)^2 = (X_{i,j}X_{i,k} - \delta_{jk})dx_j dx_k$$

Using that $X_i = x_i + u_i$

$$(dS)^2 - (ds)^2 = ((x_i + u_i)_{,j}(x_i + u_i)_{,k} - \delta_{jk})dx_j dx_k$$

We know that $dx_{i,j} = \delta_{ij}$

$$(dS)^2 - (ds)^2 = ((\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) - \delta_{jk})dx_j dx_k$$

We multiply out the bracket

$$(dS)^2 - (ds)^2 = (u_{j,k} + u_{k,j} + u_{i,j}u_{i,k})dx_j dx_k$$

We simply define $\varepsilon_{jk} = \frac{1}{2}(u_{j,k} + u_{k,j} + u_{i,j}u_{i,k})$ and get

$$(dS)^2 - (ds)^2 = 2\varepsilon_{jk}dx_j dx_k$$

This is the lagranian approach. The next lecture very much picks up where this ended, so im going to go straight to homework in this one

5.3 Homework and examples

Example in the main part of the 1D case.

5.3.1 HW 3.1 (a)

The components of a displacement field are

$$\begin{aligned} u_x &= (x^2 + 20) \times 10^{-4} \\ u_y &= 2yz \times 10^{-3} \\ u_z &= (z^2 - xy) \times 10^{-3} \end{aligned}$$

(a) Consider the two points in the undeformed system, (2,5,7) and (3,8,9). Find the change in distance between these points.
The distance before was

$$d_{before} = \sqrt{(3-2)^2 + (8-5)^2 + (9-7)^2} = 3.7417$$

The new points are calculated using the displacement field.

$$p_{1,new} = \begin{bmatrix} 2 + (2^2 + 20) \cdot 10^{-4} \\ 5 + 2 \cdot 5 \cdot 7 \cdot 10^{-3} \\ 7 + (7^2 - 2 \cdot 5) \cdot 10^{-3} \end{bmatrix} = \begin{bmatrix} 2.0024 \\ 5.07 \\ 7.039 \end{bmatrix}$$

$$p_{2,new} = \begin{bmatrix} 3 + (3^2 + 20) \cdot 10^{-4} \\ 8 + 2 \cdot 8 \cdot 9 \cdot 10^{-3} \\ 9 + (9^2 - 3 \cdot 8) \cdot 10^{-3} \end{bmatrix} = \begin{bmatrix} 3.0029 \\ 8.144 \\ 9.057 \end{bmatrix}$$

Now the new distance is calculated

$$d_{after} = \sqrt{(3.0029 - 2.0024)^2 + (8.144 - 5.07)^2 + (9.057 - 7.039)^2} = 3.81$$

The difference is then

$$\Delta d = 3.81 - 3.7417 = 68.3 \times 10^{-3}$$

6 Lecture 6: Strain part II

Bullet points for this lecture includes

- Physical Interpretaion of the strain tensor
- Principal strains
- (self: Continuation of strain tensor)

6.1 The strain tensor

6.1.1 From Lecture 5

From last time we arrived at the Lagrangian strain tensor

$$\varepsilon_{ij}^L = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

This was done my deriving it in the undeformed coordinate basis x_i. If one on the other hand does it in the deformed coordinate basis, then we get the eularian strain tensor - only difference is the sign of the non-linear part.

$$\varepsilon_{ij}^E = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})$$

The core differences are the sign of the nonlinear terms AND the basis you take the derivative with respect to! For the Lagrangian, its with respect to x_j, for for the Eulerian its wrt X_j.

6.1.2 The small strain tensor

In the even that $u_{i,j} << 1$, then the nonlinear terms $u_{k,j}u_{k,j}$ are neglible. In other words, since u is a vector field, this is the same as saying $\nabla u << 1$. In that case the $\varepsilon_{ij}^L \approx \varepsilon_{ij}^E$ and the components of the strain tensor are reduced to (for small strains only!

$$\boxed{\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})}$$

In matrix notation

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla u + \nabla u^T)$$

We can see from pure definition, that the strain tensor is symmetric.

6.2 Notes on why its not just as in the 1D case - (and rotation) - A LOT OF FORMULAS HERE ARE CONCEPTUAL

In 1D we know that

$$\varepsilon = \frac{\partial u}{\partial x}$$

So a natural thought in 3D would be that the strain tensor would be defined as $\boldsymbol{\varepsilon} = \nabla u$ - but THIS IS NOT THE CASE! This leads to some very non-physical behavior. Lets first consider a displacment vector field u. u contains information about strain, rigid body translational motion, and rigid body rotation. Lets first entertain the idea that $\boldsymbol{\varepsilon} = \nabla u$. If u is purely rigid body translation, I.E something like

$$u = \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix}$$

Then $\nabla u = 0$, which is correct. If we instead consider the case of pure rigid body rotation around the x_2 axis, such that

$$u = \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}$$

Then $\nabla u \neq 0$. In fact

$$\nabla u = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

This shows that the displacement gradient does not only contain information about deformation of matter (strain), but also contains information of rotation - hence we cannot use it as the definition of the strain tensor. To get the strain, we need to somehow take the displacement gradient and remove any contribution from rotation - this is possible. We take a small detour into the maths.

A tensor A may be split up into its symmetric part and skew symmetric part. The defintions are as follows:

$$A_{sym} = \frac{1}{2}(A + A^T)$$
$$A_{skew} = \frac{1}{2}(A - A^T)$$

In indicinal notation this is simply

$$A_{ij,sym} = \frac{1}{2}(A_{ij} + A_{ji})$$
$$A_{ij,skew} = \frac{1}{2}(A_{ij} - A_{ji})$$

It holds that

$$A = A_{sym} + A_{skew}$$

We may use this. It turns out that that the rotational part of the displacement gradient is directly linked to the rotational - so by subtracting it, we get rid of the rotational parts. We do it as follows

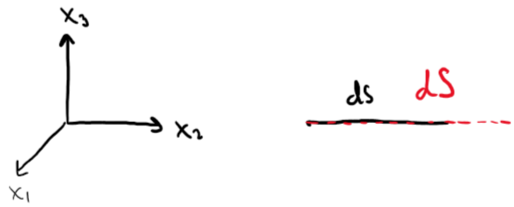
$$\boldsymbol{\varepsilon} = \nabla u - \frac{1}{2}(\nabla u - \nabla u^T) = \frac{1}{2}(\nabla u + \nabla u^T)$$

And this is the correct formulation of the strain tensor!

6.3 Physical interpretation of the Strain Tensor

6.3.1 Extensional strains (normal strains)

Lets first look at a special case, where we consider a fiber parallel to the x_2-axis of initial length ds and final length dS. For this fiber, $dx_1 = 0, dx_2 = ds, dx_3 = 0$. (this is due to the definition of arc



length really, like $ds = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$. A figure is shown of this configuration From before we derrived that $(dS)^2(ds)^2 = 2\varepsilon_{jk}dx_jdx_k$. In this configuration we would then get

$$(dS)^2 - (ds)^2 = 2\varepsilon_{22}dx_2dx_2 = 2\varepsilon_{22}(ds)^2$$

Using he difference of two squares and rewriting a bit

$$(dS - ds)(dS + ds) = 2\varepsilon_{22}(ds)^2$$

$$\Updownarrow$$

$$\frac{dS - ds}{ds} = \frac{2ds}{dS + ds}\varepsilon_{22} = \frac{ds + ds}{dS + ds}\varepsilon_{22}$$

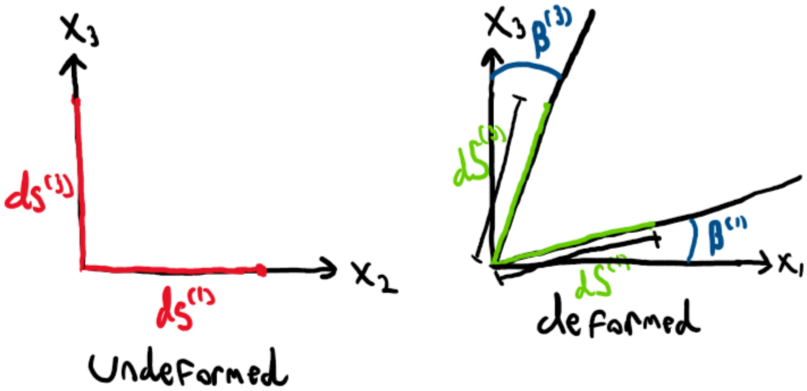
Now since the change is super small, $\frac{ds+ds}{dS+ds} \approx 1$. This yields

$$\varepsilon_{22} \approx \frac{dS - ds}{ds}$$

This means that ε_{22} is the relative elongation of a fiber parralel to the x_2-axis. Similar interpretations can be done for ε_{11} & ε_{33}. These entries sit on the diagonal of the strain tensor, and are the relative elongation in that direction.

6.3.2 Shear strains

We focus on the components $\varepsilon_{ij}, i \neq j$. We consider two fibers initially parallel to two of the coordinate axes. In this case, we consider fibers initially parralel to the x_1 and x_3 axes.



It is apparent that $\beta^{(1)} + \beta^{(3)} = \beta$. From lengthy derivation, the book arrives at

$$\beta = (u_{1,3} + u_{3,1}) = 2\varepsilon_{13}$$

That is the strain component ε_{13} is equal to on half in the increase in the angle between two fibers. (important note: Shearing stresses are commonly taken as the total angle increase, this is engineering strain, but this violates the tensor character). In **pure shear**, then β

6.3.3 Rotation vs Shear strain

Building upon the example from before. We split up the displacement gradient into two parts

$$u_{i,j} = \varepsilon_{ij} + \omega_{ij}$$

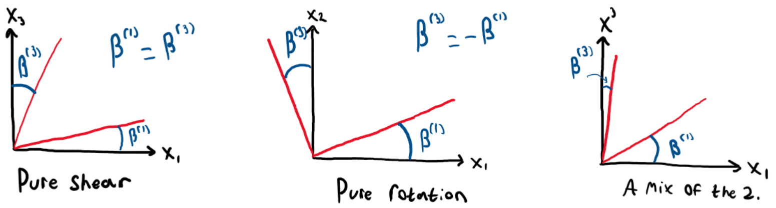
ω_{ij} is the skew symmetric part of the displacement gradient. The components are defined as

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

Before we arrived at $\beta = 2\varepsilon_{13}$. If we compare then we may state

$$\omega_{13} = \frac{\beta^{(1)} - \beta^{(3)}}{2}$$

Which is the average rotation of all fibers around the x_2-axis. For a rigid body rotation $\beta^{(1)} = -\beta^{(3)}$ which results in $\varepsilon_{13} = 0$ and $\omega_{ij} = \beta^{(1)}$. Some examples are provided below:



Further notes are

- For pure shear $u_{1,3} = u_{3,1}$
- For pure rotation $u_{1,3} = -u_{3,1}$

6.4 More on rotation

The rotation tensor is given as the skew symmetric part of the gradient

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

This tells us about the average rotation around axes. So for example ω_{13} tells about the average rotation around the x_2 axis.

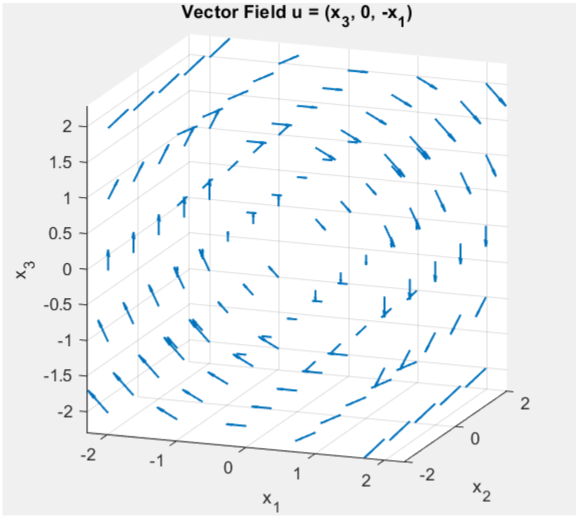


Figure 8: Rigid body rotation about x_2-axis

6.5 Principal strain

We saw that the strain tensor exhibits the same behavior as the stress tensor, this means that the principal strains are the eigenvalues and the directions are the eigenvectors of the strain tensor. I.E. solving

$$\boxed{(\varepsilon_{ij} - \lambda\delta_{ij})n_j = 0}$$

Or in tensor formulation

$$(\boldsymbol{\varepsilon} - \lambda \boldsymbol{I})\mathbf{n} = \mathbf{0}$$

An important note is that the principal directions of strain do not generally coincide with the principal directions of stress. They do coincide in the case of isotropic elastic materials.

In principal directions, we have pure normal strain, so no shear.

7 Lecture 7: Volume change & comparability

The bullet points for this lecture are

- Volume and shape change
- Compatability equations
- Summary of tensor calculus, algebra, stresses and strains
- Constituve equations (Kommer i lec 8)
- Introduction to generalized Hooke's law (Kommer i lec 8)

7.1 Volume and shape change

It is convenient to seperate the components of strain into those that causes change in volume, and those that cause changes in shape.

We consider a volume element oriented in the principal directions. The shearing stresses are then equal to 0. The original side of the lengths are $dx^{(k)}$. We may calculate the final length as

$$dX^{(k)} = dx^{(k)}(1 + \varepsilon^{(k)})$$

Thus we may find the original volume as

$$dv = \prod_{k=1}^{3} dx^{(k)}$$

Then the final volume is

$$dV = \prod_{k=1}^{3} dX^{(k)} = \prod_{k=1}^{3}(1 + \varepsilon^{(k)})dx^{(k)} = dv \prod_{k=1}^{3}(1 + \varepsilon^{(k)})$$

We may define the **dilatation** as the relative change in volume

$$\Delta = \frac{dV - dv}{dv} = \frac{dv(\prod_{k=1}^{3}(1 + \varepsilon^{(k)}) - 1)}{dv} = [(1+\varepsilon^{(1)})(1+\varepsilon^{(2)})(1+\varepsilon^{(3)})]-1 = \varepsilon^{(1)}+\varepsilon^{(2)}+\varepsilon^{(3)}+\text{higher order terms}$$

Ultimately we may approximate the change in volume by

$$\Delta = \sum_{k=1}^{3} \varepsilon^{(k)}$$

From this is pretty apparent that the elements on the diagonal is what contributes to change in volume.

If we wish to know about the change in shape instead, we have to look at the shearing strains $\varepsilon_{ij}, i \neq j$. These are called destrutions. We may decompose the strain tensor into its two components

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_M + \boldsymbol{\varepsilon}_D$$

In which

$$\boldsymbol{\varepsilon}_M = \begin{bmatrix} \varepsilon_M & 0 & 0 \\ 0 & \varepsilon_M & 0 \\ 0 & 0 & \varepsilon_M \end{bmatrix} = \text{mean normal strain}$$

And

$$\boldsymbol{\varepsilon}_D = \begin{bmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_M \end{bmatrix} = \text{Strain Deviator}$$

and where

$$\varepsilon_m = \frac{1}{3}\varepsilon_{ii}$$

The deviatoric component characterizes a change in shape of an element with no change in volume.

For stress its analogous, the stresses are also associated with the volume and shape change in the same manner. We define

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_M + \boldsymbol{\sigma}_D$$

Where

$$\begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$$

And

$$\boldsymbol{\sigma_D} = \begin{bmatrix} \sigma_{11} - \sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_M \end{bmatrix}$$

Again where

$$\sigma_M = \frac{1}{3}\sigma_{ii}$$

7.2 Compatibility

We see that we may compute the components of strain if the displacement is known from $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ (provided they are once differentiable) - HOWEVER! The inverse problem is not so direct, since we have six independent equations and only three unknowns, so the problem is harder. If we just write down six functions and call them "strain-components", they may not correspond to any real deformation, the body might "tear" or "overlap" if you tried to reconstruct the displacement field from these random strain components. For that reason we have the compatability equations.

The compatability equations are a set of conditions that must be satisfied by the strain components in order for them to correspond to something physically possible. One example could be the St. Venant compatibility equations.

7.3 Recap until now

8 Lecture 8: Constitutive equation & Linear isotropic materials and the uniaxial stress.

8.1 Constitutive equations

8.1.1 0D example

We'll start this section by considering 1D example. We consider a 1D linear spring.

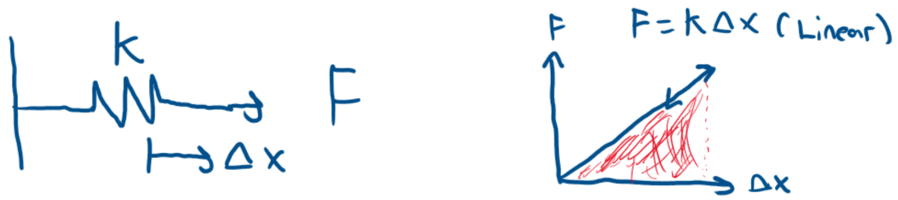


Figure 9: Displacement of a spring

This is an example of linear elasticity. The strain energy in such a case is the area under the graph.

$$u_e = \frac{1}{2}F\Delta x = \frac{1}{2}k(\Delta x)^2$$

Thus the strain energy in a linear material is quadratic. On integral form then

$$u_e = \int_0^{\Delta x} F dx = \int_0^{\Delta x} k\Delta x dx$$

This also indicates that

$$F = \frac{\partial u_e}{\partial \Delta x}$$

Hookes law: We know the 1D Hookes law, which states in that in linear deformation, the more you pull the more you strain, so for the spring this would mean

$$F = \Delta x \varepsilon$$

We need deformation to have forces, thus we need deformation to have stresses. A more general 1D linear elastic Hooke's law is

$$\sigma = E\varepsilon$$

8.1.2 Back to 3D

3D is not as nice as 2D - we don't just have 1 σ and 1ε, but entire tensors. We thus stand here with the problem

$$\sigma_{ij} = ??\varepsilon_{ij}$$

We want to relate all the strains to all the stresses, the must intuitive way is by using a fourth order tensor

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$$

This is the 3D equivalent of hookes law, here instead of having one spring stiffness, as in $F = \Delta x$, E_{ijkl} contains $3^4 = 81$ components. We may write this in matrix notation as

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}.$$

Imagine a 3D lattice where all points are connected by springs, one can think of E_{ijkl} as individual spring constants. When you twist and turn the lattice, the stiffness of each of the springs connecting to the points are going to change. If we return to the elastic energy in 3D, it would be something like

$$u_e = \frac{1}{2}(E_{ijkl}\varepsilon_{ij}\varepsilon_{kl})$$

Where in the parts $E_{ijkl}\varepsilon_{kl} = \sigma_{ij}$. This is also quadratic due to the epsilon terms, to in general, if we want force to be linear, then we need energy to be quadratic.

We may state the following. Since the stress is the derivative of the energy with respect to the strain

$$\frac{\partial u_e}{\partial \varepsilon_{ij}} = \sigma_{ij}$$

Which also implies?

$$\frac{\partial^2 u_e}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = E_{ijkl} = E_{klij}$$

So we may swap the pairs of indices, this is the major symmetry of the elasticity tensor. Therefore we go from 81 to 36 independent components. We can reduce further using the fact that

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} = \sigma_{ji} = E_{jikl}\varepsilon_{kl}$$

This implies that $E_{ijkl} = E_{jikl}$, which implies minor symmetry. We may reduce further from 36 to 21 independent variables. So an important message is **To describe the constitutive relationship for an anisotropic material we need 21 material parameters.**

8.1.3 3D for isotropic materials

This simplifies life a lot, since the material does not care on which way we pull on it - thus we only need 2 independent material parameters. The law for isotropic materials are

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

Or inverted

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\delta_{ij}\sigma_{kk}$$

We may put this on a matrix equation using voigt notation

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{Bmatrix}$$

Some conclusions can be drawn. The meaning of the first minor (so the first quadrant) - its the normal part, it connects normal stresses with only normal strains, thus it must contain both youngs modulus and poisons ratio. The fourth minor (fourth quardrant) is the shear part, it connects shear stresses with only shear stresses. This is a way of seeing, that normal stresses only impact normal strains, and shear stresses only impact shear strains. So if you want shear strain, you will never get it by applying ONLY normal forces. These λ and μ are the lame constants, and they are perfectly suitable, but we often define them in terms of other engineering constants. If we consider the situation where we only have stress in σ_{11} and all other stresses are 0, we would then define the youngs modolus

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$$

And the possoins ratio as

$$-\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} = \frac{\lambda}{2(\mu+\lambda)} = \nu$$

For λ to remain finite, it must hold that

$$-1 < \nu < 0.5$$

A physical interpretation is, that if ν is positive, then the material contracts in the other directions, when pulled in one direction. If its 0 it does not contract or expand, and if its negative, it expands in the other direction. In the case of $\nu = 0.5$ the relative change in volume is 0 and the material is called incompressible.

If we want to do it in terms of the engineering constants, we may write it as

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}$$

Where $\mathbf{C}^{-1}$ is the called compliance matrix. In Voigt notation

$$\left\{\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{array}\right\} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{array}\right\}$$

8.1.4 Small note about transverse isotropic

For a transverse isotropic material, we have 5 constants instead of 2.

9 Lecture 9: Thermal Strains, the elasticity problem, the de saint venant problem

The bullet points from this lecture are

- Thermal Strains
- The elasticity problem
- The De Saint Venant problem (This is in lecture 10 instead)
- Examples / Exercises

9.1 Thermal strains

Temperature changes may be a source of stresses in elastic system. In this course, we assume a linear relationship between the thermal strain and the temperature

$$\varepsilon_{(ii)}(T) = \alpha[T - T_0]$$

I dont know why the book only says its for the normal strains, but oh well. - This is because it causes contraction and expansion, but it doesnt really change angles. Thus we may include thermal strains in our generalized hookes law as

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right] + \alpha\delta_{ij}[T - T_0].$$

If the thermal strains cause tension, the material fractures, if it causes compression then it might buckle.

9.2 The elasticity problem

The elasticity problem, is the problem of how solid bodes deform and develop internal stresses in response to external forces or displacements - under the assumption that the material behaves elastically.

Usually we seek to find the displacement field $\mathbf{u}(x_1, x_2, x_3)$. If this is found we can compute both strain field and tensor field. The primary unknowns are the elements of the displacement vector

- $u_1(x_1, x_2, x_3)$
- $u_2(x_1, x_2, x_3)$
- $u_3(x_1, x_2, x_3)$

The governing equations are

- The kinematic equation: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ (PDE)
- The constituve eqution $\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$ (Algebraic eq)
- Equilibrium $\sigma_{ij,j} + f_i = 0$ (2nd order PDE - strain er den afledte af u, og nu afleder vi stress igen som er direkte relateret til strain)

This constitutes a total of 15 equations. Usually some boundary conditions are also to be met

- Dirichlett BC: Boundary conditions on the displacement field directly. Something like $u_3(z = L) = 0$.
- Neumann BC: Boundary conditions on stress. Maybe the surface is stress free.
- Dirichlett BC: Boundary conditions on rotation may also occur ω. (in a similar manner neumann conditions may occur on applied moment, but we have not gone thorugh any of this in the course)

10 Lecture 10: The De Saint Venant Problem

In this lecture the bullet points are

- The De Saint Venant Problem
- Simple Bending
- The Torsion Problem
- Torsion of thin walled sections: hydrodynamic analogy

10.1 The De Saint Venant Problem

10.1.1 Assumptions and introduction

De Saint Venant considered straight cylinders

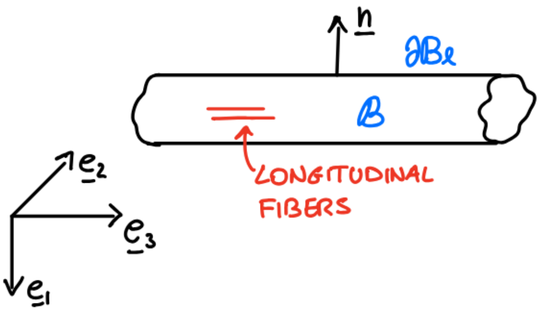


Figure 10: Straight Cylinder

Where

- ∂B_l is the lateral surface (the surface excluding the base surfaces)
- B is the body / volume of the straight cylinder

The main assumptions for the De Saint Venant Problems are

- $\underline{\mathbf{n}} \cdot \underline{\mathbf{e_3}} = 0$ on the lateral surface ∂B_l. This mean that $\underline{\mathbf{n}}$ is always perpendicular to ∂B_l
- Loads can only act on base surfaces
- Longitudinal fibers can only exchange tangential stress. This implies that $\underline{\underline{\sigma}}\ \underline{\mathbf{n}} \cdot \underline{\mathbf{n}} = 0, \forall \underline{\mathbf{n}} : \underline{\mathbf{n}} \cdot \underline{\mathbf{e_3}} = 0$. This means that normal stresses between neighboring longitudinal fibers are not allowed - only shear stresses can exist between them.

The last assumption can be written as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} = 0 \tag{2}$$

An explanation of each term is as follows:

- The vector with only n_1 and n_2 components is a unit normal vector lying entirely in the cross-sectional plane of the cylinder: Its perpindicular to $\underline{\mathbf{e_3}}$. This means that the plane it is representing is the plane between neighbouring longitudinal fibers (which run along the cylinder axis)
- The first part $\underline{\underline{\sigma}} \cdot \underline{\mathbf{n}}$ is the traction vector on this plane. Its the stress vector acting on the plane with this normal
- The second multiplication (taking the dot product with $\underline{\mathbf{n}}$ again) gives us the normal component of this traction vector. This is what we just stated had to be zero.

Upon doing the matrix multiplication we get that

$$\sigma_{11} n_1^2 + 2\sigma_{12} n_1 n_2 + \sigma_{22} n_2 = 0, \forall n_1, n_2 \tag{3}$$

Since the equation has to hold for any n_1, n_2, this leads to the result that:

$$\boxed{\sigma_{11} = \sigma_{22} = \sigma_{12} = 0} \tag{4}$$

As a consequence, the cauchy stress tensor simplifies to

$$\begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} = 0 \tag{5}$$

Where σ_{33} is the only normal stress present, and the rest are the shear stresses.

10.1.2 Equilbrium equations

We now consider a general plane of normal $\underline{\mathbf{n}}$

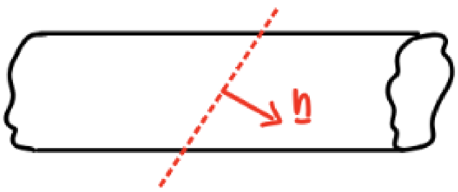


Figure 11: Cylinder cut via general plane of normal $\underline{\mathbf{n}}$

If the cylinder is cut via an arbitrairy plane $\underline{\mathbf{n}}$, we may calculate the traction vector on this plane as

$$\underline{\mathbf{T}_{\mathbf{n}}} = \underline{\underline{\sigma}}\,\underline{\mathbf{n}} = \begin{pmatrix} \sigma_{31}n_3 \\ \sigma_{32}n_3 \\ \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{pmatrix} = \begin{pmatrix} \sigma_{31}n_3 \\ \sigma_{32}n_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{pmatrix} \tag{6}$$

Now moving to the Equilibrium equations. Without body forces, the equations are $\mathbf{div}\underline{\underline{\sigma}} = 0$. Within the De Saint Venant assumptions, they become

- $\sigma_{31,3} = 0$
- $\sigma_{32,3} = 0$
- $\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0$

We define

$$\underline{\tau}_3 = \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \end{pmatrix}$$

And rewrite the equations as

- $\underline{\tau}_{3,3} = 0$
- $\mathbf{div}\underline{\tau}_3 = -\sigma_{33,3}$

10.1.3 Boundary conditions on the lateral surface

We have to enforce that there is stress on the boundary surface ∂B_l. (as there are no external forces acting here) Mathematically, this may be stated as $\underline{\underline{\sigma}} \cdot \underline{\mathbf{n}} = 0$ on ∂B_l where $\underline{\mathbf{n}} \cdot \underline{\mathbf{e_3}}$. This yields that

$$\begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} = \sigma_{31} n_1 + \sigma_{32} n_2 = 0 \tag{7}$$

Which then impllies

$$\tau_3 \cdot \mathbf{n} = 0$$

So the shear stress is tangent to the boundary. Not sure what he wanted to achieve with this? Afterwards he computes traction and stresse

10.1.4 The De Saint Venant Principle

Statically equivalent force systems, acting on the base surfaces of straight cylinders, give rise to the same stress/strain states, except for small regions near the base surfaces.

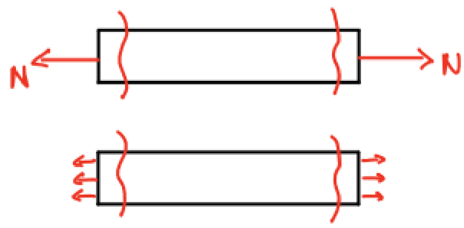


Figure 12: Statically Equivalent force systems

10.1.5 Bary center of a cross section

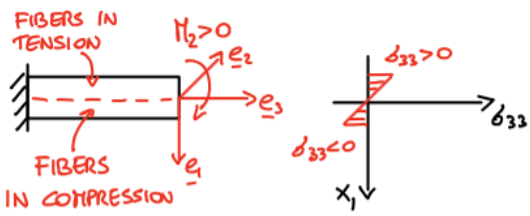
Define a local reference frame $(0; \mathbf{e}_1, \mathbf{e}_2)$ Then the position vector of G in this reference frame can be compued as

$$\mathbf{x}_G = \int_A \mathbf{x} dA$$

Where $\mathbf{x}_G$ is the vector from origin to the barycenter.

10.2 Simple bending

We consider for an example, the case where $M_2 \neq 0$ is the only torque acting on the body.



And Since the shear forces and the torsional moment are zero, we have $\boldsymbol{\tau}_3 = 0, \sigma_{33} \neq 0$. To satisfy combatability, σ_{33} has to be linear with x_1, x_2. Moreover since $M_1 = 0$, σ_{33} cannot vary with x_2 (compatability reduces to a 2D laplace eq in this case, idk, has to be linear therefore). Also since $M_1 = 0$, then the σ_{33} cannot vary with x_2 (nothing to change the stress in that direction), so we may state that

$$\sigma_{33} = \alpha x_1$$

We wish to find the moment, we integrate the force times distance over the area (if that makes sense)

$$M_2 = -\int_A \sigma_{33} x dA = -\alpha \underbrace{\int_A x^2 dA}_{\textbf{Mom of iner}} = -\alpha I_2$$

I_2 denotes moment of inertia around $\mathbf{e}_2$. This yields that

$$\sigma_{33}(x_1) = -\frac{M_2}{I_2} x_1$$

The problem is linear, so we may do the same stuff when considering only normal forces, or the other moment. Superposition for the general case yields

$$\sigma_{33}(x_1, x_2) = \frac{N}{A} - \frac{M_2}{I_2} x_1 + \frac{M_1}{I_1} x_2$$

10.3 Torsion

We consider circular cross sections (Both full and thin-walled). The analysis of circular cross section is easier since they don't warp under torsion.

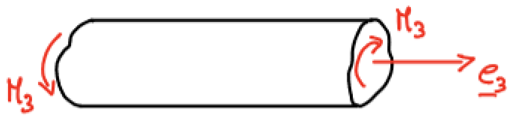


Figure 13: Circular cross section under torsion

Each section rotates around $\mathbf{e}_3$ by $\theta(x_3)$. The rotation is linear, i.e

$$\theta(x_3) = \underbrace{\theta'(x_3)}_{\text{Const}} x_3$$

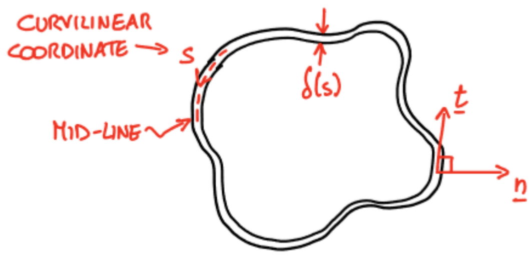
He derived a bunch of stuff but ends up with defiing the torsional rigidity as

$$C_T = GJ = \frac{M_3}{\theta'}$$

Now this is for a solid cross section!

10.3.1 Thin walled sections

A thin walled session may look like



Where

- $\delta(s)$: Thickness
- L: Length of the midline
- $\delta << L$
- $s \in [0, L]$
- $n \in [-\frac{\delta}{2}, \frac{\delta}{2}]$

There is no stress no the boundary, we have to enforce that. He uses this cool stuff to derive the breds formulas. The first bredt formula is related to the shear stress in the z direction is given as

$$\tau_z(s) = \frac{M_t}{2A_m\delta(s)}$$

On top of that we have the second bredt formula, which talks about the torsional rigidity

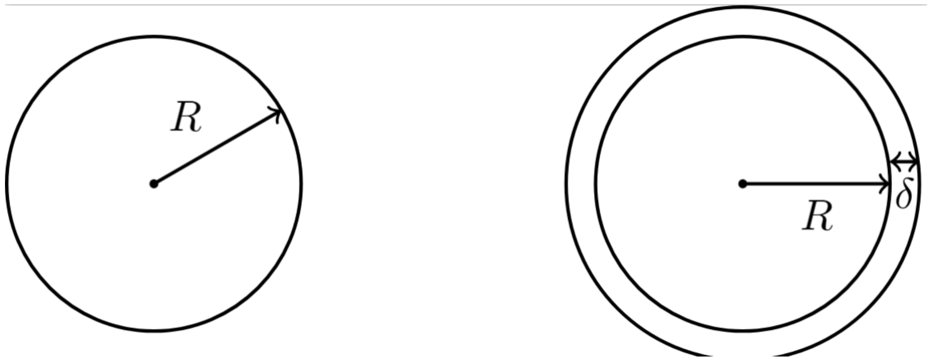
$$C_T = GJ = \frac{4GA_m^2}{\int \frac{ds}{\delta(s)}}$$

- G = shear modulus
- A_m = area enclosed by midline
- M_t = Torsional moment

10.3.2 Thin walled vs thick walled: Example

We may consider a thin walled and a thick solid cylinder to compare their torsional stifness. For the solid cylinder, the torsional stiffness is given as

$$GJ = G\frac{1}{4}\pi R^4 \propto GR^4$$



Thus a 10% increase in R causes a 40% increase in torsional stiffness. If we instead examine the hollow cylinder (using the approximation that $\int \frac{ds}{\delta(s)} \approx \frac{R}{\delta}$ we get

$$GJ = \frac{4GA_m^2}{\int \frac{ds}{\delta(s)}} \approx \frac{4GR^4}{\frac{R}{\delta}} \propto G\delta R^4 = GR^4(\frac{\delta}{R})$$

So we see that the thin walled section is weaker by the solid shaft in torsion of a factor $\frac{\delta}{R}$ - this is called is slenderness.

11 Example appendix

11.1 Stress

11.1.1 Plane stress

When we don't have any stress in one of our directions, then we have plane stress.

11.2 Strain

11.2.1 Max shear

Max shear is 45 degrees from principal strain (think 2D). The formula is

$$\tau_{max} = \frac{\sigma^{(3)} - \sigma^{(1)}}{2}$$

In these cases the maximum shear are the off diagonal elements, and the diagonal elements are the average normal stress. Something like

$$\varepsilon = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

11.2.2 Pure shear

In the case of pure shear, the matrix might look like

$$\varepsilon = \begin{bmatrix} 0 & \varepsilon_{12} \\ \varepsilon_{12} & 0 \end{bmatrix}$$

There can only be pure shear if the trace is 0!

11.2.3 Going from principal stress from max shear

Consider a case we're we have normal strain, so we have our principal configuration

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{(1)} & 0 \\ 0 & \varepsilon_{(2)} \end{bmatrix}$$

If i want a configuration with max shear, (in 2D) the diagonal elements will be the mean of the two diagonal elements - and the off diagonal elements will be the distance between them divied by two, so

$$\varepsilon = \begin{bmatrix} \frac{\varepsilon_{(1)}+\varepsilon_{(2)}}{2} & \frac{\varepsilon_{(2)}-\varepsilon_{(1)}}{2} \\ \frac{\varepsilon_{(2)}-\varepsilon_{(1)}}{2} & \frac{\varepsilon_{(1)}+\varepsilon_{(2)}}{2} \end{bmatrix}$$

Always where of course $\varepsilon_{(2)} > \varepsilon_{(1)}$

11.3 Constitutive equations

11.4 Physical interpretations

12 Spørgsmål

12.1 Spørgsmål 1:

What is dilatation? How is it defined?

12.1.1 Svar

Dilatation is the relative change of volume. An approximation is $\Delta = \varepsilon_{ii}$

12.2 Spørgsmål 2:

Where does the eigenvalue problem come from? (Physical into mathematical)

12.2.1 Svar

Derive that bitch.

12.3 Spørgsmål 3:

What are the compatability equations (i dont want you to write them, but just explain why they are there, and their use)

12.3.1 Svar

We see that we may compute the components of strain if the displacement is known from $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ (provided they are once differentiable) - HOWEVER! The inverse problem is not so direct, since we have six independent equations and only three unknowns, so the problem is harder. If we just write down six functions and call them "strain-components", they may not correspond to any real deformation, the body might "tear" or "overlap" if you tried to reconstruct the displacement field from these random strain components. For that reson we have the compatability equations. The compatability equations are a set of conditions that must be satisfied by the strain components in order for them to correspond to something physically possible.

12.4 Spørgsmål 4:

Write Gauss divergernce theorem. In index notation and in direct notation

12.4.1 Svar

Direct:

$$\int_V \nabla \cdot \mathbf{G} dV = \int_A (\mathbf{G} \cdot \mathbf{n}) dA$$

Index:

$$\int_V G_{i,i} dV = \int_A G_i n_i dA$$

12.5 Spørgsmål 5

What are the eigenvalues and eigenvectors of

$$\sigma = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

12.5.1 Svar

$$[\lambda_i]_{i=1}^3 = [p, p, 0]$$

Eigenvector for λ_3 is $(0,0,1)$. The ones for $\lambda_1 \& \lambda_2$ are any linear combination of $(1,0,0)^T$ and $(0,1,0)^T$.

12.6 Spørgsmål 6

Given plane stress, is there also guaranteed plane strain? Give physical arguments or mathematical if you can.

12.6.1 Answer

No, there is no guarantee. The poission ratio can induce strain in other directions than those of the applied stresses. It can be seen directly from the generalized hookes law

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$$

12.7 Spørgsmål 7

Can you come up with real life examples for the following

- (a) Hydrostatic stress
- (b) Plane stress
- (c) Uniaxial stress

Answer

- (a) A body submerged in water
- (b) A thin metal sheet or plate
- (c) Suspension bridge cable, a concrete column under load (approximately)

12.8 Spørgsmål 8:

What are the advantages of the airy stress function? When is it used?

12.8.1 Svar

The airy stress function is used when plane stress or strain may be assumed. It is powerful because it satisfies the equilibrium equations by construction, and it reduces compatibility to a single PDE. This it may be solved

12.9 Spørgsmål 9:

What are the boundary conditions of a coffee cup sitting on a table?

12.9.1 Answer

- Vertical displacement = 0 (cannot move through the table)
- IF no sliding, then horizontal displacements are also 0
- The table provides a reaction force, thus the stress on the bottom surface is the same as the pressure applied on the contact area (for the bottom only)
- The side and top surfaces are on the outside are stress free, this on these surfaces $\sigma_{ij}n_j = 0$ where n_j is the outward normal at the surface.
- Inside the cup (if liquid is present) then the inner wall has a small hydrostatic pressure, but this is most likely neglible.
- No strain at the top (although strain is usually not imposed as a boundary condition)

12.10 Spørgsmål 10:

Which stresses give rise to strains in other directions than their own ones? (For linear isotropic materials)

12.10.1 Svar

The normal stresses. Can be seen on generalized Hookes law. effect.

12.11 Spørgsmål 11:

Hvordan måler man youngs modulus?

12.11.1 Svar

Apply a known stress, measure the strain. Divide stress by strain (or take the slope of the stress-strain curves linear part)

12.12 Spørgsmål 12:

Hvordan måler man shear modulus?

12.12.1 Svar

Apply a known shear stress, then measure the angle, and divide shear stress by angle.

12.13 Spørgsmål 13:

Can i get shear strains if i only apply a normal stress? (for a isotropic material)

12.13.1 Svar

Næh.

12.14 Spørgsmål 14:

If $\nu = 0.5$, what is the dilatation of the material? What is this a material of this type called?

12.14.1 Svar

The dilatation is 0, the material is called incompressible.

12.15 Spørgsmål 15:

In the equation

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$$

What is the name of the matrix $\mathbf{C}$ called? And what of its inverse $\mathbf{C^{-1}}$

12.16 Spørgsmål 16:

What type of boundary conditions are associated with the displacement field? What type of boundary conditions are associated with stress?

12.16.1 Svar

One has to recall that our displacement field is in this case what we treat as our "unkown"

- Displacement: Dirichlet Boundary conditions. We enforce values at boundaries
- Stress: Neumann, since σ_{ij} is a function of displacement gradients (via strain)

12.17 Spørgsmål 17:

Can you ALWAYS find a direction of pure shear?

12.17.1 Svar

No. If the trace is not zero, then there will never be a case where all diagonal elements are 0.

12.18 Spørgsmål 18:

If i increase the radius of a cylinder (not hollow, but a solid cylinder) by 10%. By approximately how much will i increase the torsional stiffness?

12.18.1 Svar

Approx 40%. Due to

$$\underbrace{GJ}_{stiffness} = G\frac{1}{4}\pi R^4$$

12.19 Spørgsmål 19:

By what factor is a thin walled cylinder approximately weaker than a full bodied one?

12.19.1 Svar

$$\frac{h}{R}$$

This is known as the slenderness. For higher radius or small h is becomes a lot weaker.

12.20 Spørgsmål 20: Er kompatibilitetsbetingelserne nødvendige men ikke tilstrækkelige?

12.20.1 Svar

Kompatibilitetsbetingelserne for strain er **nødvendige**, men **ikke tilstrækkelige** alene for eksistensen af et displacement-felt. Dette skyldes følgende:

- **Nødvendighed:** Et vilkårligt strain-felt ε_{ij} kan *kun* stamme fra et forskydningsfelt u_i, hvis det opfylder kompatibilitetsbetingelserne:

  $$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

  og

  $$\text{Kompatibilitetsbetingelser (3D):} \quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

  (Bemærk: Ovenstående er den generelle form i 3D.)

- **Ikke tilstrækkelighed:** Selv hvis kompatibilitetsbetingelserne er opfyldt, er det ikke altid muligt at finde et displacement-felt, før *passende randbetingelser* også er opfyldt. Derudover er løsningen kun entydig op til en stiv legemebevægelse (rigid body motion).

Konklusion:

Kompatibilitetsbetingelserne er nødvendige, men ikke tilstrækkelige for eksistensen af et displacement-felt; der skal også tages højde for randbetingelser og eventuelle stive legemebevægelser.